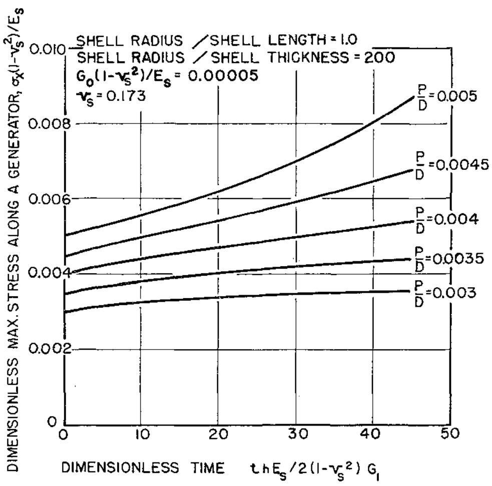

Fig. 1 Maximum stress along any shell generator vs time.

$\nu_c = \frac{1}{2}$ and replacing the shear modulus of the elastic core, $E_c/2(1 + \nu_c)$, with the shear modulus operator of (1):

$$p(w) = -\sum_{m=1}^{\infty} 2f(\lambda)\left[\left(\frac{G_0 w_m}{a}\right) + \frac{G_1 \partial(w_m/a)}{\partial t}\right]\sin\left(\frac{\lambda x}{a}\right) \quad (7a)$$

$$\lambda = m\pi a/L \quad (7b)$$

$$w = \sum_{m-1}^{\infty} w_m \sin\left(\frac{\lambda x}{a}\right) \quad (7c)$$

$$f(\lambda) = \left[\frac{\lambda I_0(\lambda)}{I_1(\lambda)}\right]^2 - 1 - \lambda^2 \quad (7d)$$

This expression for $p(w)$ and its Fourier series expansion of w is substituted into (6), giving a linear, first-order, ordinary, differential equation for w_m. It would be a simple matter to express w_m as an integral if desired:

$$(w_m/a)[\lambda^4 k - \lambda^2(2\nu_s k + P/D) + 1 - \nu_s^2 + 2f(\lambda)aG_0/D] + (2aG_1/D)d(w_m/a)dt = \begin{cases} 0 \text{ if } m \text{ is even} \\ 4a\nu_s P/D\lambda L \text{ if } m \text{ is odd} \end{cases} \quad (8)$$

The boundary condition (4b) is satisfied by (6). The Fourier series expansion of w satisfies (4a). The initial condition (4c) will be satisfied if

$$w_m = 0 \text{ at } t = 0 \quad (9)$$

Expressions (7d, 8, and 9) give the deformations, and these, in turn, can be used to calculate the stress σ_x at the outside or inside surface of the shell:

$$\alpha_x = \frac{-P}{D} \pm \frac{(3k)^{1/2}w''}{a} = \frac{-P}{D} \pm (3k)^{1/2}\sum_{m=1}^{\infty}\left(\frac{\lambda^2 w_m}{a}\right)\sin\left(\frac{\lambda x}{a}\right) \quad (10)$$

This stress was computed for an example problem, the parameters given below being used. For simplicity, only the largest stress σ_x was computed:

$$\mathrm{P}(t) = \begin{cases} 0 \text{ for } t < 0 \\ P_0 \text{ for } t \geq 0 \end{cases}$$

$$G_0(1 - \nu_s^2)/E_s = 0.00005 \qquad \nu_s = 0.1732$$

$$a/L = 0.5 \qquad a/h = 200$$

For a load P_0 below a certain critical value, the stress α_x approaches a limiting value as time increased (i.e., as $t \rightarrow + \infty$). For loads above this critical value, the stresses increase exponentially with time. The increase is more rapid with larger loads (see Fig. 1).

The critical load is the "long-time" buckling load of the case. If loads were held constant for a long period of time, one could neglect the viscoelastic characteristics of the core and treat it as elastic body. The buckling or Euler load of a case with this elastic core would be the "critical load" just mentioned.

The solid viscoelastic core used in this calculation stiffened the case considerably, and the results are not generally applicable to a practical situation. It is recognized that the more realistic case would have been for a hollow cylindrical core; however, the method of incorporating the viscoelastic model has been demonstrated as was the intent of this note. A core with a concentric hole can be incorporated by defining a new $f(\lambda)$. The arduous details of this change can be found in Seide's paper.[1]

References

[1] Seide, P., "Stability under axial compression and lateral pressure of circular cylindrical shells with soft elastic core," J. Aerospace Sci. **29**, 851–862 (1962).

[2] Yao, J. C., "Buckling of axially compressed long cylindrical shell with elastic core," J. Appl. Mech. **29**, 329–334 (1962).

[3] Flugge, W., *Stresses in Shells* (Springer-Verlag, Berlin, 1960), Chap. 3, p. 130 and Chap. 7, p. 422.

One-Dimensional Flow Considering Buoyancy Forces

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Nomenclature

T = absolute temperature
C_p = specific heat at constant temperature
C_v = specific heat at constant volume
γ = ratio of the specific heats
P_γ = Prandtl number
g = acceleration due to gravity

Subscripts

0 = condition at $x = 0$
α = conditions at $x \rightarrow \infty$
μ = coefficient of viscosity of the fluid
β = coefficient of volume expansion
K = thermal conductivity of the fluid

1. Introduction

An exact solution of nonlinear equations for viscous and heat conducting compressible fluid has been given by Morduchow and Libby[1], Pai[2], and Ludford[3] for one-dimensional flow. They neglected buoyancy forces; but in motions where temperature differences change density, it becomes necessary to include buoyancy forces in the equation of motion of a viscous fluid, and they should be treated as impressed body forces. The buoyancy forces are caused by changes in volume which are associated with the temperature differences. Now, if β is the coefficient of volume expansion, and if $\theta = T - T_\infty$ is the temperature difference between a hotter fluid particle and the colder surroundings, the change in volume per unit

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volume of the hotter particle is $\beta\theta$, and so lift force per unit volume is $\rho g\beta\theta$, where ρ denotes the density at temperature T and g the acceleration due to gravity. In the present note, we shall show that a solution may also be obtained, taking account of the buoyancy forces.

One may think of the motion of a heated gas upward across a plane, which we may take as the $x = 0$ plane, where the temperature is T_0 and there is a given mass of flow of definite amount upward per unit time. The temperature as the gas moves upward cools down to a limiting temperature T_∞, where the velocity is u_∞ for complete solution viscosity and conductivity have been taken as constants, and the Prandtl number has been taken as $\frac{3}{4}$, which is not far from the correct value for air.

2. Governing Equations

We take the axis vertically upward. The motion is supposed to be entirely in this direction and is governed by the equations

$$\frac{d}{dx}(\rho u) = 0 \tag{1}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} - \frac{4}{3}\frac{d}{dx}\left(\mu \frac{du}{dx}\right) - \rho g \beta (T - T_\infty) = 0 \tag{2}$$

$$\rho u \frac{d}{dx}(C_v T) + p\frac{du}{dx} - \frac{d}{dx}\left(K\frac{dT}{dx}\right) - \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 = 0 \tag{3}$$

To this we add the equation of state in the form

$$p = R\rho T \tag{4}$$

These equations are to be solved under the boundary conditions

$$T = T_0,\ \rho = \rho_0,\ u = u_0 \quad \text{at} \quad x = 0 \tag{5}$$

$$T \to T_\infty \quad \text{as} \quad x \to \infty \tag{6}$$

3. Solutions

Equation (1) gives a constant flux m across every plane x = const as, say,

$$\rho u = \rho_0 u_0 = m \tag{7}$$

Thus, using (4) and (7), one has

$$p = mRT/u \tag{8}$$

$$\frac{dp}{dx} = \frac{mR}{u}\frac{dT}{dx} - \frac{p}{u}\frac{du}{dx} \tag{9}$$

Replacing dp/dx in (2) by (9), one has

$$\frac{p}{m}\frac{du}{dx} = u\frac{du}{dx} + R\frac{dT}{dx} - \frac{4}{3}\frac{u}{m}\frac{d}{dx}\left(\mu\frac{du}{dx}\right) - \beta g(T - T_\infty) \tag{10}$$

Replacing the second term on the left of (3) by its value in (10), one gets an equation that may be expressed as

$$\frac{d}{dx}\left(C_pT + \frac{u^2}{2}\right) - \frac{4}{m}\frac{d}{dx}\left(\mu u\frac{du}{dx}\right) - g\beta(T - T_\infty) - \frac{1}{m}\frac{d}{dx}\left(K\frac{dT}{dx}\right) = 0$$

which, upon integration, gives

$$\frac{u^2}{2} + C_pT - \frac{K}{mc_p}\left[\frac{4}{3}\frac{\mu C_p}{K}\frac{d}{dx}\left(\frac{u^2}{2}\right) + C_p\frac{dT}{dx}\right] = \text{const} + g\beta\int(T - T_\infty)\,dx$$

Now, we take as trial

$$T - T_\infty = \xi = Ae^{-\lambda x} \tag{11}$$

where λ and A are constants, and $P_r = \frac{3}{4}$. The preceding equation then becomes

$$\frac{d}{dx}\left(\frac{u^2}{2} + C_pT\right) - \frac{mcp}{K}\left(\frac{u^2}{2} + C_pT\right) = \text{const} + \frac{g\beta mC_p}{\lambda K}Ae^{-\lambda x}$$

Upon integration, assuming K = const,

$$\frac{u^2}{2} + C_pT = \text{const} - \frac{g\beta C_p m}{\lambda K[\lambda + (mC_p/K)]} A\exp(-\lambda x) + C_2\exp\left(\frac{mC_p}{K}\right)x$$

where C_2 is a constant of integration.

Let us assume the flow to be uniform at $+\infty$ so that C_2 is zero; hence the equation becomes

$$u^2/2 + C_pT = \text{const} - g\beta/\lambda/[1 + (\lambda K/C_p m)]\xi$$

whence

$$(u^2/2) + D\xi = \text{const.}$$

where

$$D = C_p + B \tag{11'}$$

As

$$x \to \infty \qquad u \to u_\infty \qquad \xi \to 0$$

we have

$$(u^2/2) + D\xi = (u_\infty^2/2) \tag{12}$$

Equation (12) is the integral of the one-dimensional energy equation of a steady flow of a viscous, compressible, heat conducting fluid, under the assumptions made. Our assumption (11), which yields the integral (12), should be compatible with the momentum equations (2) and (3) to insure that we proceed as follows:

$$udu/dx = D\lambda\xi \tag{13}$$

Then, from (8), one has

$$\frac{dp}{dx} = Rm\left[\frac{dT}{dx}\cdot\frac{1}{u} - \frac{T}{u^2}\frac{du}{dx}\right] = -R\rho\left[\lambda\xi + \frac{(T_\infty + \xi)\lambda D\xi}{u_\infty^2 - 2D\xi}\right] \tag{14}$$

From (13), differentiating with respect to X, and arranging,

$$\frac{ud^2u}{dx^2} = -\lambda^2 D\xi - \frac{\lambda^2D^2\xi^2}{[u_\infty^2 - 2D\xi]} \tag{15}$$

We substitute the values of the derivatives so obtained in Eqs. (2) and (3), and obtain the following two equations:

$$[3mu_\infty^2(D\lambda - g\beta) - 3Rm(\lambda u_\infty^2 + T_\infty\lambda D) + 4\mu\lambda^2u_\infty^2D] - D\xi[6m(D\lambda - g\beta) - 3Rm\lambda + 4\lambda^2\mu D] = 0 \tag{16}$$

$$[3RmT_\infty D - 3u_\infty^2(mc_v + K\lambda)] - D\xi[4\mu\lambda D - 6(mC_v + K\lambda) - 3Rm] = 0 \tag{17}$$

In order that (11) may be compatible with (2) and (3), (16) and (17) should hold good for all values of ξ. This requires that we should have the following four relations:

$$3mu_\infty^2(D\lambda - g\beta) - 3Rm(\lambda u_\infty^2 + T_\infty\lambda D) + 4\mu\lambda^2Du_\infty^2 = 0 \tag{18}$$

$$3m[\lambda(2D - R) - 2g\beta] + 4\mu\lambda^2D = 0 \tag{19}$$

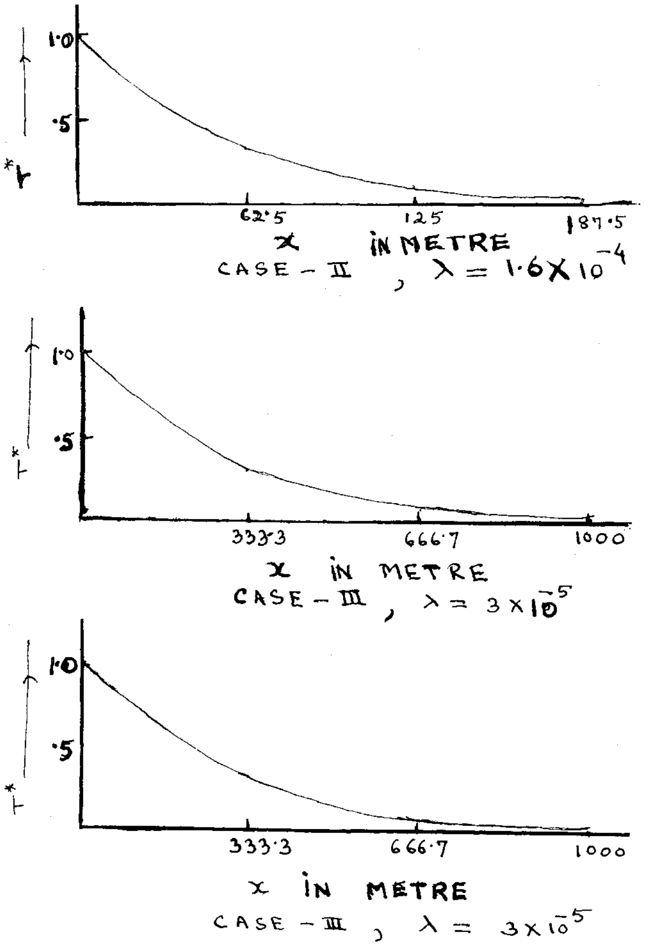

Fig. 1 Graphs for T_∞ for different values of λ.

$$RmT_\infty D - u_\infty^2(mC_v + K\lambda) = 0 \quad (20)$$

$$4\mu\lambda D - 6(mC_v + K\lambda) - 3Rm = 0 \quad (21)$$

All four equations (18–21) are not, however, independent. We may note that in obtaining these equations we have differentiated the energy equation (12), which, however, has been deduced from the two momentum equations and the equation of continuity. It may be shown that two identical relations can be obtained from the four preceding equations so that only two of them are independent. To show this, we eliminate D between (20) and (21) and obtain

$$3RmT_\infty\{2(mC_v + K\lambda) + Rm\} - 4\mu\lambda u_\infty^2(mC_v + K\lambda) = 0 \quad (22)$$

In the same way, eliminating D between (18) and (21), we obtain

$$3RmT_\infty\{2(mC_v + K\lambda) + Rm\} = [-4\mu m(R\lambda + g\beta) + (3m + 4\mu\lambda)\{2(mC_v + K\lambda) + Rm\}]u_\infty^2 \quad (23)$$

If now, from (22) and (23), we eliminate the ratio $T_\infty : u_\infty^2$ the resulting equation that follows is

$$3m^2\ [2C_v + R) + m\ [6K\lambda + 4\mu\lambda C_v - 4\mu g\beta] + 4\mu K\lambda^2 = 0 \quad (24)$$

The elimination of D between (19) and (21) gives, exactly, Eq. (24). Thus, one relation among the preceding four equations is established. Again, if we substitute the value of D from (11) in (21), and take the difference, the resultant equation and $C_p \times$ (24), we obtain

$$m[3\lambda K(2C_v + R) + 6K\lambda C_p - 4\mu\lambda C_p^2 - 4\mu\lambda C_pC_v] - 4\mu C_p\lambda^2 K + 6K^2\lambda^2 = 0$$

This equation is found to be identically satisfied when in this we replace K from our assumption $C_p\mu/K = \frac{3}{4}$. This is the second relation.

Thus, only two of the four equations (18–21) are independent, and we take (22) and (24) to be these independent equations.

From (24), eliminating K with the help of $C_p\mu/K = \frac{3}{4}$, we obtain the following equations for λ:

$$16\mu^2\gamma\lambda^2 + 12\mu m[2\gamma + 1]\lambda + 9m^2(\gamma + 1) - 12m\mu g\beta/C_v = 0 \quad (25)$$

λ should be real and positive. The discriminant of (25) is

$$144\mu^2m^2[(2C_p + C_v)^2 - 4C_p(C_p + C_v)] + 788\mu^3C_pg\beta$$

for

$$C_p = 0.2375 \text{ cal/g-deg and } C_v = 0.1684 \text{ cal/g-deg}$$

The discriminant is positive so that the real root exists.

Now, in Eq. (25), the coefficient of λ is positive, and if the constant term is negative, there will be one positive root.

The constant term will be negative if, say,

$$m < \frac{4}{3}\cdot\frac{\mu g\beta}{C_v(\gamma + 1)} = \alpha \quad (26)$$

When $\gamma = C_p/C_v$, solving (25), the positive root of λ is given by

$$\lambda = \frac{-3m(\gamma + 1) + 3m(\gamma + 1)\ (1 + \eta)^{1/2}}{8\mu\gamma} \quad (27)$$

when

$$\eta = \frac{4\gamma(\gamma + 1)}{m(2\gamma + 1)^2}(\alpha - m)$$

In the practical problem of atmospheric convection, we shall be interested in small values of λ so that η should be a small positive quantity; under this condition,

$$\lambda \simeq \frac{3}{4}\cdot\frac{\gamma + 1}{\mu(2\gamma + 1)}\epsilon \text{ when } \epsilon = \alpha - m \quad (28)$$

For this it is evident that the present problem of atmospheric convection will have the solution (11), namely,

$$T - T_\infty = \xi = (T_0 - T_\infty)e^{-\lambda x}$$

The exponential decrease of temperature with height is familiar in meteorological discussions, but the present result represents a steady-state dynamical situation with no idea of initial isothermal or adiabatic equilibrium of the atmospheric column.

The preceding result is, however, meaningful when the value of m is slightly below that of α, since, in that case, the value T_∞ will be approached at considerable heights above $x = 0$. λ is determined by (27), and then one obtains T_∞ and u_∞ from (12) and (22), which yield

$$\frac{u_0^2}{2} + D(T_0 - T_\infty) = \frac{u_\infty^2}{2}$$

and

$$u_\infty^2 = \frac{\{(u_0^2/2D) + T_0\}[2(mC_v + K\lambda) + Rm]}{\{(2mC_v + K\lambda) + Rm\}/2D + (mC_v + K\lambda) \times (4\mu\lambda/3Rm)}$$

The distribution of T and u are given by (11) and (12). Now

$$\lambda = \frac{3(\lambda + 1)}{4\mu(2\gamma + 1)}\left[\frac{\alpha}{\rho_0} - u_0\right]$$

For exponential decrease of temperature to be valid, $u_0 < \alpha/\rho_0$ as u_0 approaches α/ρ_0, λ becomes smaller and smaller, which is desirable for atmospheric studies.

Table 1 Table for T_∞ for different values of λ

ρ	u_0	I_0, deg	λ	μ_∞	T_∞
1.13×10^{-3}	1.539	313	3×10^{-3}	0.34×10^3	264°2
1.13×10^{-3}	1.5393	313	1.6×10^{-4}	1.47×10^3	264°1
1.13×10^{-3}	1.53934	313	3×10^{-5}	3.6×10^3	258°0

For air, $\mu = 1.64 \times 10^{-4}$, $\beta = \frac{1}{273}$, $C_v = 0.1689$, $C_p = 0.2375$, $\gamma = 1.66$, $\rho = 1.13 \times 10^{-3}$ approximately at 40°C., and $g = 981$; therefore, $\alpha = 1.74 \times 10^{-3}$. Since m should be $< \alpha$, i.e., $u_0 < 1.53935$ cm/sec., then $\lambda = 3.15(1.53935 - u_0)$; since a small value of λ is desirable, u_0 should be very close to 1.53935, but should not exceed it. Therefore, the initial motion at $x = 0$ admissible under this solution is very slow. For a different value of u_0, T_0 and $\rho_0 = 1.3 \times 10^{-3}$. The values of u_∞ and T_∞ are given in Table 1.

Now $T^* = [(T - T_\infty)/(T_0 - T_\infty)]$, and T^* is plotted against x for different values of λ in case I, case II, and case III (see Fig. 1).

At higher temperature, ρ is much smaller, and we can take u_0 greater than that in the preceding case. Also, if we consider a column of gas on a planet or a star where g is large and ρ is small, u_0 can be taken much larger.

References

[1] Morduchow, M. and Libby, P. A., "On a complete solution of the one-dimensional flow equations of a viscous, heat-conducting compressible fluid," J. Aeronaut. Sci. **16**, 674–684 (1949).

[2] Pai, S. I., *Viscous Flow Theory–I—Laminar Flow* (D. Van Nostrand Co., Inc., New York, 1956).

[3] Ludford, G. S. S., "The classification of one-dimensional flows and the general shock problem of a compressible, viscous, heat conducting fluid," J. Aeronaut. Sci. **18**, 830–834 (1951).

Delays in Initiation of Discharges in Pulsed Plasma Accelerators

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PULSED plasma accelerators used in electrical propulsion investigations are operated by the charging of a capacitor bank, and then rapidly discharging the bank through a propellant gas that is contained between a pair of coaxial or nozzle shaped electrodes. The gas, which is initially un-ionized, is quickly converted by the discharge to a highly ionized plasma and is ejected with high velocity from the accelerator by MHD forces. In some of these devices[1, 2] the capacitor bank is permanently connected to the electrodes. The accelerators operate in a vacuum chamber that is maintained at sufficiently low pressure so that no discharge occurs between the electrodes while the bank is charging. When the bank is fully charged, a puff of propellant is injected between the electrodes by very briefly opening a gas valve, which raises the pressure sufficiently to initiate the discharge. The efficiency of propellant utilization in these accelerators can be markedly affected by the time interval between the injection of the gas and the time when the discharge current becomes large enough to accelerate the gas. If too long a time elapses, propellant may leak out of the accelerator into the surrounding vacuum and be wasted. An interval of even a few hundred microseconds may be sufficient to cause appreciable loss.

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It is often assumed that the discharge will commence as soon as the pressure in a region between the electrodes is raised by the puff of gas so that the d.c. sparking potential in that region becomes equal to or slightly less than the interelectrode voltage. The subsequent rate of current rise is considered to be determined solely by the circuit inductance and capacitance. These assumptions neglect several factors that could lengthen the time required to reach a high current level. Among these are statistical time lag,[3] which is the time required for a free electron to enter the gas. Since the propellant is initially un-ionized, it will remain a practically perfect insulator until at least one free electron enters it. The ionizing collisions made by this electron produce additional charge carriers that initiate the electrical breakdown. Statistical lags can be relatively long, depending on the strength of the sources supplying free electrons to the discharge region. Generally, radioactive or other sources of free electrons are not used in pulsed plasma accelerators, and the free electron must originate in an accidental manner.

After initiation of breakdown, current density grows to large values, densities exceeding 10^7 amp/m^2 being common.[2] To support these large densities, the cathode must emit electrons copiously. It may take an appreciable time after initial breakdown for the physical mechanisms that provide this emission to become operative. This time would be an additional source of delay before current could grow to large magnitudes.

To investigate the magnitude of the delay in a high-current discharge, breakdowns between two plane parallel circular aluminum electrodes in low-pressure stagnant nitrogen gas were observed. The electrodes were 1 in. in diameter and were separated by a 1-cm gap. A thyratron switch connected a charged 1-μf capacitor bank across the electrodes when the thyratron was triggered. The electrode voltage was measured with a voltage probe and discharge current was measured with a Rogowski coil. When the thyratron was triggered, the voltage oscilloscope traces showed that the capacitor voltage was applied across the electrodes with a rise time less than 0.1 μsec. The voltage remained constant until a high current discharge occurred. The discharge produced a sharp decrease in voltage which coincided with the beginning of the damped sinusoidal discharge current signal. The peak value of the discharge current was almost 2000 amp. All the reported measurements were taken with an initial capacitor voltage of 3 kv, the maximum voltage rating of the capacitor bank. Lower voltages did not consistently produce discharges in the pressure range of 0.2 to 1.0 mm at which the measurements were made. The measured value of the d.c. sparking potential in this pressure range was about 350 v. The interval between successive discharges was generally 30 sec. No noticeable changes were produced when this interval was increased to 5 min.

The delays found in 139 measurements at a pressure of 0.2 mm Hg and 60 measurements at 0.7 mm Hg are shown in Fig. 1. The fraction of the delays falling in 40-μsec intervals is plotted. There was considerable scatter in the measured values. Examination of the data shows that half the delays at 0.2 mm Hg were longer than 160 μsec and more than 20% were longer than 480 μsec. At 0.7 mm Hg, the delays were somewhat shorter, half being longer than 80 μsec and 10% longer than 480 μsec. Delays of several milliseconds were occasionally observed. Thus, a considerable fraction of the observed delays were long enough to affect propellant efficiency.